Differences between quantum and classical adiabatic evolution

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Adiabatic evolution is an emergent design principle for time modulated metamaterials, often inspired by insights from topological quantum computing such as braiding operations. However, the pursuit of classical adiabatic metamaterials is rooted in the assumption that classical and quantum adiabatic evolution are equivalent. We show that this is only true in the limit where the frequencies of all the bands are at infinite distance from 0 and some instances of quantum adiabatic evolution, such as those containing zero modes, cannot be reproduced in classical systems. This is because mode coupling is fundamentally different in classical mechanics. We derive classical conditions to ensure adiabaticity and demonstrate that only under these conditions—which are different from quantum adiabatic conditions—do the single band Berry phase and Wilczek-Zee matrix for everywhere degenerate bands emerge as meaningful quantities encoding the geometry of classical adiabatic evolution. Finally, for general multiband systems we uncover a correction term in the non-Abelian gauge potential for classical systems.

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I. INTRODUCTION

Quantum-classical analogies have grown into a vibrant research domain, particularly with the emergence of topological photonic [1] and phononic [2] metamaterials. More recently, classical metamaterials with time and space modulation [3–5] achieve topological pumping [6–15] and non-Abelian physics [16–18], inspiring both fundamental research and novel information processing concepts. At the heart of these advancements lies quantum adiabatic evolution and the corresponding geometric quantities [19–24]. Until now, studies have assumed that classical adiabatic evolution is subject to gap conditions analogous to those in the quantum case and the geometric quantities arising are equivalent [3,6,9,14–16,25–29]. Our article challenges this assumption by demonstrating fundamental differences in the mode-coupling equations arising from the fact that a time-dependent Hermitian classical system is mapped to a time-dependent non-Hermitian quantum system. This has two consequences: (i) it introduces a correction term in the non-Abelian gauge potential and (ii) it implies an additional condition to ensure adiabaticity in classical systems, rendering concepts such as braiding of zero modes untenable. Nevertheless, we identify a limit where these differences vanish and classical adiabatic evolution approaches the quantum case.

A. Quantum adiabatic evolution

In the context of a slowly changing quantum Hamiltonian $H^q(t)$ with instantaneous eigenstates $|n(t)\rangle$ and instantaneous eigenvalues $\lambda_n(t)$, a quantum state initially prepared as $|\psi^q(0)\rangle = |n(0)\rangle$, and evolving under the Schrödinger equation $i|\dot{\psi}^q\rangle = H^q(t)|\psi^q\rangle$, remains in the nth eigenstate, approximately described as $|\psi^q_{\text{adia}}(t)\rangle \approx \exp[-i\Lambda(t) - \gamma(t)]|n(t)\rangle$ [19]. Here, $\Lambda_n(t) := \int_0^t dt'\lambda_n(t')$ represents the dynamic phase and $\gamma_n(t) = \int_0^t dt'\langle n(t')|\dot{n}(t')\rangle$ corresponds to the geometric phase, known as the Berry phase. In what follows we choose a parallel transport gauge, i.e., $\langle n(t)|\dot{n}(t)\rangle = 0$. Then the Berry phase can be computed as $\gamma(t) = \text{Im ln}\,\langle n(0)|n(t)\rangle$ [30,31]. To obtain $|\psi^q_{\text{adia}}(t)\rangle$, we expand $|\psi^q(t)\rangle$ in terms of the instantaneous eigenstates as $|\psi^q(t)\rangle = \sum_n q_m(t)\exp[-i\Lambda_m(t)]|m(t)\rangle$. Substituting this into the Schrödinger equation and contracting with $\langle n|$, we integrate from 0 to t, leading to [32–35]

$$q_n(t) - q_n(0) = -\sum_{m\neq n}\int_0^t dt' q_m(t')\langle n|\dot{m}\rangle e^{-i\int_0^{t'} dt''\lambda_{mn}(t'')}, \tag{1}$$

where $q_n(0) = \delta_{nm}$ and $\lambda_{mn}(t) = \lambda_m(t) - \lambda_n(t)$. The right-hand side (RHS) can be interpreted as the error of the adiabatic approximation due to the excitation of other modes and is negligible if the exponential in the integrand oscillates rapidly compared to the slowly changing other terms. This condition is captured by the quantitative gap condition,

$$\max_{t\in[0,T_f]}\left|\frac{\langle n|\dot{m}\rangle}{\lambda_{mn}(t)}\right| \ll 1 \quad \forall n \neq m, \tag{2}$$

where T_f is the final time of integration.

B. Classical adiabatic evolution

In this article we consider a classical evolution where a dynamical matrix $H^c(t)$ is changed slowly. $H^c(t)$ is

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symmetric, positive definite, and real-valued as classical systems have real-valued couplings, but in principle the reasoning is also valid for complex-valued Hermitian matrices. The dynamics of the classical state vector $|\psi^c(t)\rangle$ is governed by the Newtonian second-order equation $|\ddot{\psi}^c\rangle = -H^c(t)|\psi^c\rangle$. We denote the real-valued instantaneous frequency of $H^c(t)$ as $\omega_n(t) := \sqrt{\lambda_n(t)} > 0$, the dynamic phase as $\Omega_n(t) := \int_0^t dt'\omega_n(t')$, and we choose the instantaneous eigenstates real-valued and in a parallel transport gauge. As we will prove shortly, for an initial condition of $|\psi^c(0)\rangle = |n(0)\rangle$ and $|\dot{\psi}^c(0)\rangle = 0$, the classical adiabatic approximation reads

$$\left|\psi^c_{\text{adia}}(t)\right\rangle \approx \sqrt{\frac{\omega_n(0)}{\omega_n(t)}}\cos[\Omega_n(t)]|n(t)\rangle. \tag{3}$$

We observe that the amplitude is not conserved and increases (decreases) as the mode becomes softer (stiffer). The Berry phase is hidden in the eigenstates and can be extracted as $\gamma(t) = \text{Im ln}\,\langle n(0)|n(t)\rangle$. For a closed-loop evolution, i.e., $H^c(T) = H^c(0)$, the Berry phase is obviously quantized to 0 or π modulo 2π, protected by the fact that H is purely real-valued and symmetric.

C. Problem statement and structure of the paper

The expression (3) has also been obtained using WKB assymptotics in [3]. However, there and in other studies [6,9,14,15,25–29] it is assumed that the classical adiabatic condition is exclusively based on a frequency gap, analogous to the quantum gap condition (2). We will show, using a quantum formalism, that a gap condition (2) is not sufficient and, more generally, quantum and classical adiabatic evolution are different except in the limit of infinitely high frequencies.

The study is based on contrasting the quantum coupling equations (1), with classical coupling equations, from which we deduce the adiabatic conditions and approximations. In Sec. II we derive the coupling equations by applying a transformation that would symmetrize the problem in the time-independent case but fails to do so for time-dependent systems. In Sec. III we explore the results in the light of biorthogonal operator theory. Then, from the coupling equations we derive classical adiabatic conditions and compare them to quantum conditions in Sec. IV. We find fundamental differences and identify a limit where they become equivalent. In particular, setting $H^c := H$ and $H^q := \sqrt{H}$, we will show that, while generally different, in the limit where the full spectrum of H is at infinite distance from 0, both the nonadiabatic and adiabatic dynamics of H^q and H^c will be the same. In Sec. V we further study energy consumption of adiabatic evolution. We then present a simple numerical example in Sec. VI. Finally, in Sec. VII we generalize our findings to the multiband, non-Abelian case and discuss the appearance of a correction term to the non-Abelian gauge potential in classical systems in comparison to quantum systems. Conclusions are drawn in Sec. VIII.

II. DERIVATION USING COORDINATE TRANSFORMATION

It is well known [2,36] that, in time-independent systems, symmetric classical equations of motion can be mapped to a Hermitian Schrödinger equation. However, as we now demonstrate, this is not the case for time-dependent problems. The resulting Schrödinger equation is non-Hermitian because the coordinate transformation does not commute with the time derivative.

To see this we first define $|\Psi^c(t)\rangle = [|\psi^c(t)\rangle, |\dot{\psi}^c(t)\rangle]^T$, such that the classical equation of motion can be transformed to first order

$$|\dot{\Psi}^c\rangle = \begin{bmatrix} 0 & \mathbb{1} \\ -H(t) & 0 \end{bmatrix}|\Psi^c\rangle. \tag{4}$$

We now introduce the transformation [2,36]

$$\mathcal{T}(t) = \begin{bmatrix} \sqrt{H(t)} & 0 \\ 0 & i\mathbb{1} \end{bmatrix}, \tag{5}$$

which would symmetrize Eq. (4), i.e., mapping it to a Hermitian Schrödinger equation, for time-independent H.

Let us further define $|\tilde{\Psi}^c\rangle := \mathcal{T}(t)|\Psi^c\rangle$ and

$$\tilde{H}(t) := \begin{bmatrix} 0 & \sqrt{H(t)} \\ \sqrt{H(t)} & 0 \end{bmatrix}. \tag{6}$$

Then, transforming Eq. (4) under $\mathcal{T}(t)$, we obtain the equation of motion for $|\tilde{\Psi}^c\rangle$ as

$$i|\dot{\tilde{\Psi}}^c\rangle = [\tilde{H}(t) + i\dot{\mathcal{T}}(t)\mathcal{T}^{-1}(t)]|\tilde{\Psi}^c\rangle. \tag{7}$$

The inverse transformation, $\mathcal{T}^{-1}$, exists because H is assumed to be positive definite. This is a non-Hermitian Schrödinger equation because

$$i\dot{\mathcal{T}}(t)\mathcal{T}^{-1}(t) = i\begin{bmatrix} \sqrt{\dot{H}(t)}\sqrt{H(t)}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

is in general non-Hermitian. All the differences between quantum and classical adiabatic evolution and mode coupling arise from this term. We now expand the general (nonadiabatic) solution in the instantaneous, adiabatic eigenbases of $\tilde{H}$, i.e.,

$$|\tilde{\Psi}^c(t)\rangle = \sum_{m,\sigma} c_{\sigma,m}(t)\sqrt{\omega_m(0)\omega_m(t)}e^{-i\sigma\Omega_m(t)}|\sigma, m(t)\rangle, \tag{9}$$

where $|\sigma, m(t)\rangle := 1/\sqrt{2}[|m(t)\rangle, \sigma|m(t)\rangle]^T$ are the orthonormal instantaneous eigenstates of $\tilde{H}(t)$ in a parallel transport gauge, i.e., $\langle m(t), \sigma|\sigma, \dot{m}(t)\rangle = \langle m(t)|\dot{m}(t)\rangle = 0$, and corresponding instantaneous eigenfrequencies are $\sigma\omega_m(t)$, with $\sigma = \pm 1$ labeling the subspace corresponding to the positive or negative eigenfrequencies [36]. While for time-independent systems these two subspaces are always decoupled (as are eigenstates in general), in the time-dependent case this is no longer true. Inserting this expansion into Eq. (7), contracting with $\langle m(t), \tau|$, and integrating from 0 to t, one arrives at the classical version of Eq. (1) (see Supplemental

Material [37])

$$c_{\tau,n}(t) - c_{\tau,n}(0) = -\sum_{m\neq n}\int_0^t dt' c_{\tau,m}(t')\langle n|\dot{m}\rangle \frac{1}{2}\sqrt{\frac{\omega_m(0)}{\omega_n(0)}}\left(\sqrt{\frac{\omega_n(t')}{\omega_m(t')}} + \sqrt{\frac{\omega_m(t')}{\omega_n(t')}}\right)e^{i\tau\Omega^-_{nm}(t')}$$
$$-\sum_{m\neq n}\int_0^t dt\, c_{-\tau,m}(t')\langle n|\dot{m}\rangle \frac{1}{2}\sqrt{\frac{\omega_m(0)}{\omega_n(0)}}\left(\sqrt{\frac{\omega_n(t')}{\omega_m(t')}} - \sqrt{\frac{\omega_m(t')}{\omega_n(t')}}\right)e^{-i(-\tau)\Omega^+_{nm}(t')}$$
$$+\int_0^t dt\, c_{-\tau,n}(t')\frac{1}{2}\frac{\dot{\omega}_n(t')}{\omega_n(t')}e^{i\tau 2\Omega_n(t')}, \tag{10}$$

where $c_{\tau,n}(0) = \delta_{nm}\delta_{\tau\sigma}\langle n(0), \tau|\mathcal{T}(0)|\Psi^c(0)\rangle/\omega_n(0)$, $\Omega^-_{nm}(t) := \Omega_n(t) - \Omega_m(t)$, and $\Omega^+_{nm}(t) := \Omega_n(t) + \Omega_m(t)$. Again, the RHS can be identified as the error to the adiabatic approximation. Clearly, if the RHS is approximately 0, so is the left-hand side, i.e., $c_{\tau,n}(t) \approx c_{\tau,n}(0)$. Substituting this result back into (9), applying the inverse transformation, i.e., $\mathcal{T}^{-1}(t)|\tilde{\Psi}^c(t)\rangle$, and taking the real part results in the adiabatic approximation (3).

Because the transformation does not commute with the time derivative operator, Hermitian classical systems fundamentally map to non-Hermitian quantum systems giving rise to differences in the (adiabatic) evolution for finite frequencies, as analyzed in Sec. IV.

III. DERIVATION USING BIORTHOGONAL FORMALISM

It is customary to treat non-Hermitian systems in the biorthogonal formalism, directly starting from Eq. (4) [38]. Now we show that such a treatment produces the same coupling equations (10) which we demonstrate to be well-defined despite the time-dependent norm ambiguity of biorthogonal basis functions.

Let us define

$$G := \begin{bmatrix} 0 & \mathbb{1} \\ -H(t) & 0 \end{bmatrix}, \tag{11}$$

such that Eq. (4) reads

$$|\dot{\Psi}^c\rangle = G|\Psi^c\rangle. \tag{12}$$

The mth instantaneous eigenvalue-eigenstate pair of $H(t)$ is denoted by $[\lambda_m(t) = \omega_m^2(t), |m(t)\rangle]$. Without loss of generality, we again choose $|m(t)\rangle$ in a parallel transport gauge such that $\langle m|\dot{m}\rangle = 0$. Let us introduce the biorthogonal left and right eigenvectors of G [39]. It is easy to check that

$$|\sigma, M(t)\rangle := \frac{1}{\sqrt{2}}\begin{bmatrix} |m(t)\rangle \\ -\sigma i\omega_m(t)|m(t)\rangle \end{bmatrix},$$
$$\beta_m(t) := -\sigma i\omega_m(t), \tag{13}$$

for $\sigma = \pm 1$ is a right eigenstate-eigenvalue pair of the non-Hermitian operator (11). Its corresponding left eigenstate is given by

$$\langle\hat{N}(t), \tau| := \frac{1}{\sqrt{2}}\left[\langle n(t)| \quad \tfrac{i\tau}{\omega_n(t)}\langle n(t)|\right], \tag{14}$$

with corresponding eigenvalue

$$\hat{\beta}_n(t) = \beta_n^*(t) := \tau i\omega_n(t), \tag{15}$$

where $*$ denotes complex conjugation. Note that because $H(t)$ is assumed to be positive definite and symmetric, we have $\lambda_m(t) > 0$ and hence $\omega_m(t) \subset \mathbb{R}^+$, implying that the instantaneous non-Hermitian system is purely oscillatory.

We further note that degeneracies in H carry over to degeneracies in G. We assume that respective eigenstates of H have been orthogonalized so that corresponding left and right eigenstates are naturally biorthogonal. The left and right eigenstates satisfy the biorthogonality relation

$$\langle\hat{N}(t), \tau|\sigma, M(t)\rangle = \frac{1}{2}\left(\langle n|m\rangle + \tau\sigma\frac{\omega_m(t)}{\omega_n(t)}\langle n|m\rangle\right) = \delta_{nm}\delta_{\sigma\tau}. \tag{16}$$

Biorthogonal eigenstates of non-Hermitian operators do not only have an arbitrary phase but also an arbitrary norm [38,40]. Hence, for any at least once differentiable $f_{\sigma,m}(t) \in \mathbb{C}$, the right f-transformed eigenstates $|\sigma, M(t)\rangle \to f_{\sigma,m}(t)|\tau, M(t)\rangle$ and left f-transformed eigenstates $\langle\hat{N}(t), \tau| \to \langle\hat{N}(t), \tau|1/f^*_{\tau,n}(t)$ also form a complete biorthogonal basis. Furthermore, because here we are interested in parallel transporting $|m\rangle$ rather than $|\tau, M(t)\rangle$, in general there is no further restriction on $f_{\sigma,m}(t)$ (see [38] for more in detail discussion).

To that end we use the following nonadiabatic ansatz:

$$|\Psi^c(t)\rangle = \sum_{m,\sigma} b_{\sigma,m}(t)e^{-i\sigma\Omega_m(t)}f_{\sigma,m}(t)|\sigma, M(t)\rangle. \tag{17}$$

We substitute the ansatz into the equations of motion (12) and contract with $\langle\hat{N}, \tau|1/f^*_{\tau,n}$, where from now on we omit to show the time dependence explicitly. We obtain (see Supplemental Material [37])

$$\dot{b}_{\tau,n}f_{\tau,n}\sqrt{\omega_n}$$
$$= -b_{\tau,n}\dot{f}_{\tau,n}\sqrt{\omega_n} - b_{\tau,n}f_{\tau,n}\frac{1}{2}\frac{\dot{\omega}_n}{\sqrt{\omega_n}}$$
$$-\sum_m b_{\tau,m}f_{\tau,m}\langle n|\dot{m}\rangle\frac{1}{2}\sqrt{\omega_n}\left(1 + \frac{\omega_m}{\omega_n}\right)e^{i\tau\Omega^-_{nm}}$$
$$-\sum_m b_{-\tau,m}f_{-\tau,m}\langle n|\dot{m}\rangle\frac{1}{2}\sqrt{\omega_n}\left(1 - \frac{\omega_m}{\omega_n}\right)e^{i\tau\Omega^+_{nm}}$$
$$+ b_{-\tau,n}f_{-\tau,n}\frac{1}{2}\frac{\dot{\omega}_n}{\sqrt{\omega_n}}e^{i2\tau\Omega_n}. \tag{18}$$

To demonstrate that the results obtained in this study are well defined we have to show that the coupling equations (18)

are independent of the arbitrary choice of $\{f_{\sigma,m}\}$. We will show that this is indeed the case, as the choice of $\{f_{\sigma,m}\}$ will just turn out to be a time-dependent scaling, which is compensated by the corresponding coefficients.

To that end we define $a_{\tau,n} := b_{\tau,n} f_{\tau,n}\sqrt{\omega_n}$ with

$$\dot{a}_{\tau,n} = \dot{b}_{\tau,n} f_{\tau,n}\sqrt{\omega_n} + b_{\tau,n} f_{\tau,n}\frac{1}{2}\frac{\dot{\omega}_n}{\sqrt{\omega_n}} + b_{\tau,n}\dot{f}_{\tau,n}\sqrt{\omega_n}. \quad (19)$$

We further use $b_{\tau,n} = a_{\tau,n}/(f^*_{\tau,n}\sqrt{\omega_n})$ and, because

$$\dot{b}_{\tau,n} f_{\tau,n}\sqrt{\omega_n} = \dot{a}_{\tau,n} - a_{\tau,n}\frac{1}{2}\frac{\dot{\omega}_n}{\omega_n} - a_{\tau,n}\frac{\dot{f}_{\tau,n}}{f^*_{\tau,n}}, \quad (20)$$

all the diagonal terms in Eq. (18) vanish and all f terms are absorbed into the coefficients $a_{\tau,n}$ such that the coupling equations for $a_{\tau,n}$ read

$$\begin{aligned}\dot{a}_{\tau,n} =& -\sum_m a_{\tau,m}\langle n|\dot{m}\rangle\frac{1}{2}\left(\sqrt{\frac{\omega_n}{\omega_m}} + \sqrt{\frac{\omega_m}{\omega_n}}\right)e^{i\tau\Omega^-_{nm}} \\ &- \sum_m a_{-\tau,m}\langle n|\dot{m}\rangle\frac{1}{2}\left(\sqrt{\frac{\omega_n}{\omega_m}} - \sqrt{\frac{\omega_m}{\omega_n}}\right)e^{i\tau\Omega^+_{nm}} \\ &+ a_{-\tau,n}\frac{1}{2}\frac{\dot{\omega}_n}{\omega_n}e^{i2\tau\Omega_n}. \end{aligned} \quad (21)$$

We note that the above equations are equivalent to (10) up to factors of $\sqrt{\omega_n(0)/\omega_m(0)}$, which can be pulled out and absorbed in the coefficients.

Therefore, upon time integration, these coupling equations for $\{a_{\tau,n}(t)\}$ are equivalent to Eq. (10) for the coefficients $\{c_{\tau,n}(t)\}$. The ambiguous scaling factors $\{f_{\sigma,m}\}$ do not appear in the coupling equation, meaning that they are well defined.

The choice of $\{f_{\sigma,m}\}$ results in a scaling of the coefficients that leave the overall, generally nonadiabatic state invariant. To see this, we observe that $\mathcal{T}(t)\sqrt{\omega_m(0)/\omega_m(t)}|\sigma, M(t)\rangle = \sqrt{\omega_m(0)\omega_m(t)}|\sigma, m(t)\rangle$. Therefore, in Sec. II we use the biorthogonal basis with $\hat{f}_{\tau,n}(t) = \hat{f}_n(t) = \sqrt{\omega_m(0)/\omega_m(t)}$ and transform it under $\mathcal{T}(t)$. Hence, if we backtransform the ansatz (9) with $\mathcal{T}^{-1}(t)$ we get

$$|\Psi^c(t)\rangle = \sum_{m,\sigma} c_{\sigma,m}(t)\sqrt{\frac{\omega_m(0)}{\omega_m(t)}}e^{-i\sigma\Omega_m(t)}|\sigma, M(t)\rangle. \quad (22)$$

Comparing this ansatz with the one chosen in (17), we find that the different coefficients are related by a scaling of $b_{\sigma,m}(t)f_{\sigma,m}(t)\sqrt{\omega_m(t)/\omega_m(0)} = a_{\sigma,m}(t)/\sqrt{\omega_m(0)} = c_{\sigma,m}(t)$. Solving for $b_{\sigma,m}$ and substituting the expression back into (17), we see that $f_{\sigma,m}$ vanishes and therefore conclude that $|\Psi^c(t)\rangle$ is independent of the choice of $\{f_{\sigma,m}\}$, as expected.

In summary, a choice of $\{f_{\sigma,m}\}$, here $\{\hat{f}_{\sigma,m}\}$, results in a unique set of coefficients. The coupling between these coefficients is independent of $\{f_{\sigma,m}\}$. Furthermore, while the time-dependent scaling of the coefficients depends on $\{f_{\sigma,m}\}$, $|\Psi^c(t)\rangle$ does not. In Sec. II, the coefficients $\{c_{\sigma,m}\}$ were chosen such that in the adiabatic limit they remain constant.

An extensive discussion about the scaling issue, which is about how to define the population of biorthogonal eigenstates, can be found in [38]. In this article we focus on the mode coupling, which is well defined, despite the ambiguity of the norm of the biorthogonal eigenstates.

IV. CLASSICAL VS QUANTUM ADIABATIC EVOLUTION: DIFFERENCES AND A LIMIT WHERE THEY MEET

We now analyze the differences between quantum and classical adiabatic evolution by comparing the RHS of the classical coupling equations (10) and (21) to the RHS of the quantum mechanical couplings (1). (i) The first RHS term in Eq. (10) resembles the quantum mechanical coupling to other modes of the same subspace τ. In the classical case, however, the square root ratio of the tracked and all the other instantaneous frequencies scales this error term. If any mode (tracked or not tracked) approaches a zero frequency, this term diverges, requiring ever slower time evolution of $H(t)$. (ii) The second sum is the error due to coupling to other modes of the other subspace, denoted with $-\tau$. It will be clear shortly that if the first term is negligibly small, so is the second term. (iii) Finally, the last term is the coupling to the same mode of the other subspace which can lead to amplitude pumping through parametric amplification. Because this last term holds simultaneously for all $c_{\tau,n}$, small transition errors to a near-zero mode, $c_{\tau,0}$, may grow over time. This is exactly the situation in Figs. 1(c) and 1(d), as discussed shortly.

Following the same logic used to derive the quantum gap condition (2), we can now establish two classical conditions ensuring that the integrals in Eq. (10) become negligible thus implying the validity of the adiabatic approximation (3):

$$\max_{t\in[0,T_f]}\left|\frac{\langle n|\dot{m}\rangle}{\omega^-_{mn}(t)}\frac{1}{2}\sqrt{\frac{\omega_m(0)}{\omega_n(0)}}\left(\sqrt{\frac{\omega_n(t)}{\omega_m(t)}} + \sqrt{\frac{\omega_m(t)}{\omega_n(t)}}\right)\right| \ll 1 \quad \forall m \neq n \quad (23)$$

and

$$\max_{t\in[0,T_f]}\left|\frac{1}{4}\frac{\dot{\omega}_n(t)}{\omega_n^2(t)}\right| \ll 1. \quad (24)$$

We stress that condition (23) implies also that the second sum in Eq. (10) is small because the absolute difference of the positive ratios is smaller than their sum and the same applies to the more rapid oscillations, i.e., $\omega^-_{nm}(t) = \omega_n(t) - \omega_m(t) < \omega^+_{nm}(t) = \omega_n(t) + \omega_m(t)$. The first condition resembles the quantum mechanical gap condition, however, scaled by the frequency ratios. The second condition is a condition on the distance from the zero frequency line.

Note that for a single degree of freedom (DOF) harmonic oscillator there is only one coupling to its mirrored mode partner, the last line in Eq. (10). In the Supplemental Material [37] we apply our study to the single DOF case in detail.

In Eqs. (10) we can identify the limit $\omega_n \to \infty$, $\forall n$ where the classical time-dependent dynamics under $H(t)$ approach the dynamics of a quantum system under $\sqrt{H(t)}$. In this limit $\sqrt{\omega_n(t)/\omega_m(t)} \to 1$, which implies that

$$\frac{1}{2}\sqrt{\frac{\omega_m(0)}{\omega_n(0)}}\left(\sqrt{\frac{\omega_n(t')}{\omega_m(t')}} + \sqrt{\frac{\omega_m(t')}{\omega_n(t')}}\right) \to 1 \quad (25)$$

and

$$\frac{1}{2}\sqrt{\frac{\omega_m(0)}{\omega_n(0)}}\left(\sqrt{\frac{\omega_n(t')}{\omega_m(t')}} - \sqrt{\frac{\omega_m(t')}{\omega_n(t')}}\right) \to 0. \quad (26)$$

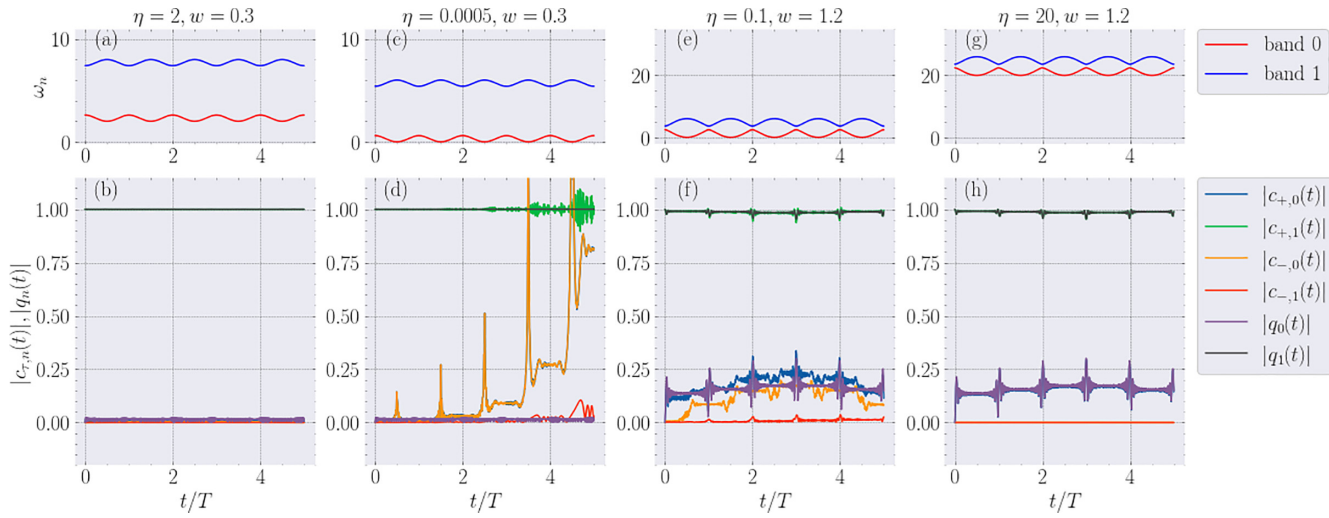

FIG. 1. Numerical simulation of a 2×2 system defined in Eq. (27), where $H^c = h^2$ and $H^q = h$. Row 1 displays the band structures, while row 2 shows the time evolution of the classical and quantum mode coefficients. The coefficients are initialized in the excited mode for various values of η and w, while $s = 3$ and $T = 10$ are fixed. (a), (b) In the presence of a large gap and the absence of a zero mode, both the classical and quantum systems exhibit adiabatic behavior. (c), (d) With a small amount of energy transitioning to the lower mode, the classical system violates adiabatic conditions due to interactions between $c_{+,0}$ and $c_{-,0}$. In contrast, the quantum system remains adiabatic. (e)–(h) The classical system approaches quantum behavior as the spectrum is shifted to higher frequencies. In (e), (f), nonadiabatic interactions occur due to the small band gap (quantum) and the combination of a small gap and a near-zero mode (classical). Shifting the spectrum to higher frequencies in (g), (h) results in the classical system where only the gap matters and the frequency ratios approach unity. Consequently, the modes from the negative frequency axis are no longer excited and the full nonadiabatic dynamics closely approximates that of the quantum system.

Finally, the last line in Eq. (10) approaches 0; therefore, the coupling dynamics reduces to the quantum case (1) and the classical adiabatic conditions to the quantum conditions. Note that this limit requires all the bands to be at sufficiently large frequencies; it does not apply to a subset of bands at high frequencies. In general, any arbitrary high frequency mode can couple to any near zero mode.

V. ENERGY CONSUMPTION

The energy consumption of adiabatic driving is nontrivial as the norm is not conserved. From the classical adiabatic approximation (3) we see that the average instantaneous energy for a system with time-independent, unit mass is given by $\langle E(t)\rangle = \frac{1}{4}\omega_n(0)\omega_n(t)$. Therefore, the adiabatic energy cost of transforming an initial state $|n(0)\rangle$ into a final state $|n(T)\rangle$ is $\Delta E = \frac{1}{4}\omega_n(0)[\omega_n(T) - \omega_n(0)]$, with $\Delta E = 0$ for a flat band. Furthermore, for a closed loop, where $\omega_n(T) = \omega_n(0)$, the total energy cost is also zero, meaning that adiabatically accumulating a Berry phase does not consume any energy.

VI. EXAMPLE

The following example illustrates the phenomena discussed above using a simple 2×2 system. To this end, we slightly reformulate the system considered in [41]. We define the following matrix:

$$h(t) := \begin{bmatrix} \eta + s + [w + (s-w)\cos(t/T + \pi)] & (s-w)\sin(t/T+\pi) \\ (s-w)\sin(t/T+\pi) & \eta + s - [w + (s-w)\cos(t/T+\pi)] \end{bmatrix}, \tag{27}$$

and identify the quantum and classical systems as $H^q = h$ and $H^c = h^2$, respectively (hence $H^q = \sqrt{H^c}$). By squaring the operator, the eigenstates and hence the Berry phase remain unchanged, while the eigenvalues are squared. This implies that the quantum and classical states oscillate with the same frequency. The parametrization of h is chosen such that changing η allows us to shift the spectrum without altering the gap. Furthermore, for $0 < w < s/2$ ($s > w > s/2$), the first (second) band has a Berry phase of π (0). The transition from 0 to π of both bands occurs at $w = s/2$, where the gap closes. Finally, T controls the slowness of integration. The parameters s and T are fixed to 3 and 10, respectively. The classical system has four modes $|+,0\rangle$, $|+,1\rangle$, $|-,0\rangle$, and $|-,1\rangle$ with corresponding coefficients $c_{+,0}$, $c_{+,1}$, $c_{-,0}$, and $c_{-,1}$. The quantum modes are $|0\rangle$ and $|1\rangle$ with coefficients q_0 and q_1. In the following, we always initialize the system in the excited mode, $|+,1\rangle$ and $|1\rangle$, respectively, setting $c_{+,1}(0) = q_1(0) = 1$ and $c_{+,0}(0) = c_{+,1}(0) = c_{-,0}(0) = c_{-,1}(0) = q_0(0) = 0$. We numerically integrate Eqs. (1) and (10) and display the norm of the coefficients over five periods in Fig. 1. In panels (a),(b), we set $\eta = 2$, $w = 0.3$. We observe that the adiabatic approximation is well justified for both the quantum (due to the large gap) and the classical (due to the large gap and no near-zero mode) system. In (c),(d), we set $\eta = 0.0005$, thereby shifting

the lower band closer to zero. This has, as expected, no effect in the quantum case, which remains adiabatic. However, in the classical case, the small amount of energy that scatters to $c_{+,0}(t)$ and $c_{-,0}(t)$ amplifies because the second condition (24) is violated for $|+,0\rangle$ and $|-,0\rangle$. Finally, panels (e)–(h) demonstrate how the classical system approaches the quantum system by shifting the spectrum to higher frequencies: in (e),(f), the parameters are chosen as $\eta = 0.1, w = 1.2$. The larger w and hence smaller gap result in a nonadiabatic transition to the lower mode. Because the lower band is still close to zero, the classical coupling to the other modes, particularly the nonvanishing coupling to the negative frequency subspace, presents a different picture in the classical than in the quantum case. As predicted, shifting the spectrum to large enough frequencies (g),(h) such that $\sqrt{\omega_0(t)/\omega_1(t)} \to 1$, the classical mode coupling behavior approaches that of the quantum system, which is, once again, left invariant by a spectrum shift.

VII. MULTIBAND CASE

We now treat the case of generally interacting bands and also reduce it to the case where a set of bands are everywhere degenerate. Let us first recall the quantum behavior: let $\mathcal{S}$ be a subspace of interacting bands that are well separated by an energy gap from all other bands such that

$$\max_{t\in[0,T_f]} \left| \frac{\langle s|\dot{m}\rangle}{\lambda_{sm}(t)} \right| \ll 1 \quad \forall s \in \mathcal{S},\ \forall m \notin \mathcal{S}. \tag{28}$$

Then Eq. (1) can be solved for the coefficients of the subspace, $\mathbf{q}_\mathcal{S}$, using the evolution operator $\mathcal{U}^q(t) := \mathcal{U}^q(t,0)$ [20,21], i.e.,

$$\mathbf{q}_\mathcal{S}(t) = \mathcal{U}^q(t)\mathbf{q}_\mathcal{S}(0), \tag{29}$$

with

$$\mathcal{U}^q(t) = \mathcal{P}\ \exp\left(-\int_0^t dt' \mathcal{Q}(t')A(t')\mathcal{Q}^{-1}(t') \right), \tag{30}$$

where $\mathcal{P}$ denotes the path ordering operator and $A_{ss'} := \langle s|\dot{s}'\rangle$ denotes the non-Abelian gauge field which is transformed under $\mathcal{Q}_{ss'}(t) := \exp[i\Lambda_s(t)]\delta_{ss'}$. In the case of everywhere degenerate bands one obtains $\mathcal{Q}(t)A(t)\mathcal{Q}^{-1}(t) \to A(t)$ and $\mathcal{U}^q(t) \to \mathcal{U}(t) = \mathcal{P}\ \exp[-\int_0^t dt' A(t')]$ is the Wilczek-Zee matrix whose trace is the gauge invariant Wilson loop [42]. To treat the classical system we denote coefficients of the particular frequency subspace, τ, corresponding to states of the subspace as $[\mathbf{c}^\tau_\mathcal{S}]_s = c_{\tau,s}$. We introduce the classical evolution operator for the coefficients, $\mathcal{U}^c(t) := \mathcal{U}^c(t,0)$, such that

$$\mathbf{c}^\tau_\mathcal{S}(t) = \mathcal{U}^c(t)\mathbf{c}^\tau_\mathcal{S}(0). \tag{31}$$

In the Supplemental Material [37] we show that

$$\mathcal{U}^c(t) = \mathcal{P}\ \exp\left(-\int_0^t dt' \mathcal{C}(t')[A(t') + A^\Delta(t')]\mathcal{C}^{-1}(t') \right), \tag{32}$$

with the classical transformation $\mathcal{C}_{ss'}(t) := \exp[i\Omega_s(t)]\delta_{ss'}/\sqrt{\omega_s(0)\omega_s(t)}$ and $A^\Delta_{ss'}(t) := \langle s|\dot{s}'\rangle(\omega_s - \omega_{s'})/(2\omega_{s'})$, which can be seen as a classical correction to the non-Abelian gauge field and again reflects the fact that the coupling between interacting modes depends on their frequencies. As shown in Sec. III the couplings and hence the classical non-Abelian gauge potential $A(t') + A^\Delta(t')$ is independent of the norm ambiguity of the biorthogonal basis. This implies that it is well defined.

The high frequency limit can be identified as $(\omega_s - \omega_{s'})/(2\omega_{s'}) \to 0$ implying that $A^\Delta_{ss'}(t) \to 0$ and $\sqrt{\omega_s(0)\omega_s(t)}/\sqrt{\omega_{s'}(0)\omega_{s'}(t)} \to 1$, implying that $\mathcal{C}(t)A(t)\mathcal{C}^{-1}(t) \to \mathcal{Q}(t)A(t)\mathcal{Q}^{-1}(t)$ and hence the classical evolution approaches the evolution of the quantum system governed by $\sqrt{H}(t)$. Substituting (31) back into the expansion (9) and applying the inverse transformation $\mathcal{T}^{-1}(t)$ yields the classical, general multiband solution

$$\psi^c_{\text{adia,mult}}(t) \approx \text{Re}\left\{ \sum_{s\in\mathcal{S}} \sum_{s'\in\mathcal{S}} \mathcal{U}^c_{ss'}(t) c_{s'}(0) \sqrt{\frac{\omega_s(0)}{\omega_s(t)}} e^{-i\tau\Omega_s(t)} |s(t)\rangle \right\}, \tag{33}$$

with initial coefficients $c_{s'}(0) = \langle s'(0), \tau|\mathcal{T}(0)|\Psi^c(0)\rangle/\omega_{s'}(0)$. It remains to state the adiabatic conditions under which the approximation is valid. The first two are a restatement of the single band case but for the whole subspace $\mathcal{S}$,

$$\max_{t\in[0,T_f]} \left| \frac{\langle s|\dot{m}\rangle}{\omega^-_{ms}(t)} \frac{1}{2} \sqrt{\frac{\omega_m(0)}{\omega_s(0)}} \left(\sqrt{\frac{\omega_s(t)}{\omega_m(t)}} + \sqrt{\frac{\omega_m(t)}{\omega_s(t)}} \right) \right| \ll 1 \quad \forall s \in \mathcal{S},\ \forall m \notin \mathcal{S} \tag{34}$$

and

$$\max_{t\in[0,T_f]} \left| \frac{1}{4} \frac{\dot{\omega}_s(t)}{\omega_s^2(t)} \right| \ll 1 \quad \forall s \in \mathcal{S}. \tag{35}$$

Finally, we have to explicitly require that the interaction of bands in the subspace is isolated from the same subspace on the negative frequency axis, $-\tau$, which is ensured if

$$\max_{t\in[0,T_f]} \left| \frac{\langle s|\dot{s}'\rangle}{\omega^+_{s's}(t)} \frac{1}{2} \sqrt{\frac{\omega_{s'}(0)}{\omega_s(0)}} \left(\sqrt{\frac{\omega_s(t)}{\omega_{s'}(t)}} - \sqrt{\frac{\omega_{s'}(t)}{\omega_s(t)}} \right) \right| \ll 1 \quad \forall s, s' \in \mathcal{S}. \tag{36}$$

In the case of everywhere degenerate bands we have $A^\Delta_{ss'}(t) = 0$ since $\omega_s = \omega_{s'}$ and $\mathcal{C}(t)A(t)\mathcal{C}^{-1}(t) \to A(t)$, which is integrated to yield, as in the quantum system, the Wilczek-Zee matrix, $\mathcal{U}(t)$. Defining $\omega_\mathcal{S}$ as the degenerate eigenfrequency of the subspace and $\Omega_\mathcal{S}$ the corresponding dynamical phase and again using that the eigenstates can always be chosen real for a real-valued symmetric system, one obtains

$$\psi^c_{\text{adia,mult,deg}}(t) \approx \frac{\sqrt{\omega_\mathcal{S}(0)}}{\sqrt{\omega_\mathcal{S}(t)}} \cos[\Omega_\mathcal{S}(t)] \sum_{s\in\mathcal{S}} \sum_{s'\in\mathcal{S}} \mathcal{U}_{ss'}(t) c_{s'}(0)_s |s(t)\rangle \tag{37}$$

and the third condition becomes trivially satisfied.

VIII. CONCLUSIONS

We have demonstrated that a time-dependent Hermitian classical system is equivalent to a time-dependent

non-Hermitian quantum system, resulting in distinct mode coupling equations. Using non-Hermitian operator theory we showed that these coupling equations are well defined.

In contrast to the time-independent case, classical time-dependent dynamics can in general not be symmetrized because the transformation does not commute with the time derivative. From this we deduced that the quantum gap condition is not sufficient to ensure adiabaticity and certain quantum adiabatic phenomena cannot be realized in classical systems. Another consequence of this is the appearance of a classical correction term to the non-Abelian gauge field for general multiband systems which deserves further investigation and may inspire a new class of time modulated metamaterials. Finally, we identified a limit where these differences vanish such that quantum and classical adiabatic and nonadiabatic dynamics become equivalent up to a difference in the dynamic phase.

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